Year 12 Mathematics Specialist 2018

Test Number 5:

Rates of Change and Differential Equations

Resource Free

Name: Solutions **Teacher: DDA**

Marks: **36**

Time Allowed: **35 minutes**

Instructions: No notes or calculators. Show your working where appropriate remembering you must show working for questions worth more than 2 marks.

Question 1 **[2, 2, 2 = 6 marks]**

Find $\frac{dy}{dx}$ if:

a) $x^4 + 5y^3 = 17$

$$4x^3 + 15y^2\frac{dy}{dx} = 0 \quad ✓$$

$$\frac{dy}{dx} = \frac{-4x^3}{15y^2} \quad ✓$$

b) $e^x + e^{2y} = x^2$

$$e^x + 2e^{2y}\frac{dy}{dx} = 2x \quad ✓$$

$$\frac{dy}{dx} = \frac{2x - e^x}{2e^{2y}} \quad ✓$$

c) $x\cos x = \sin y$

$$1\cos x + (-\sin x)x = \cos y\frac{dy}{dx} \quad ✓$$

$$\frac{dy}{dx} = \frac{\cos x - x.\sin x}{\cos y} \quad ✓$$

Question 2 **[3, 6 = 9 marks]**

Solve the following differential equations expressing your answers in the form $y = f(x)$

a) $\frac{dy}{dx} = 2y^4$

$$\int y^{-4} dy = \int 2dx$$

$$\frac{y^{-3}}{-3} = 2x + c$$

$$y^3 = \frac{1}{d - 6x}$$

$$y = \frac{1}{\sqrt[3]{d - 6x}}$$

b) $\frac{dy}{dx} = \frac{xy - 6y}{x^2 - 4}$ given that when $y = 1, x = 3.$

$$\frac{dy}{dx} = \frac{y(x-6)}{x^2 - 4}$$

$$\int \frac{1}{y} dy = \int \frac{x-6}{x^2-4} dx$$

$$\int \frac{1}{y} dy = \int \left(\frac{-1}{x-2} + \frac{2}{x+2}\right) dx$$

$$\ln|y| = -\ln|x-2| + 2\ln|x+2| + c$$

$$\ln 1 = -\ln 1 + 2\ln 5 + c$$

$$c = -2\ln 5$$

$$\ln|y| = \ln|x-2| + 2\ln|x+2| - 2\ln 5$$

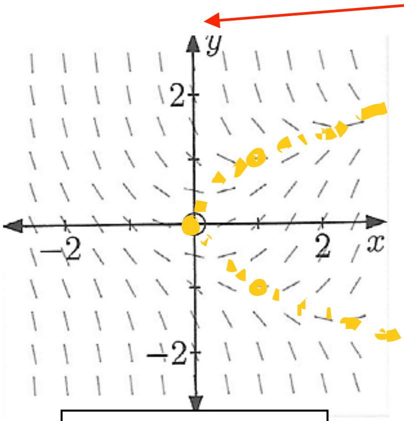
$$\ln|y| = \ln\left|\frac{(x+2)^2}{25(x-2)}\right|$$

$$y = \frac{(x+2)^2}{25(x-2)} \qquad (\,or \quad y = -\frac{(x+2)^2}{25(x-2)})$$

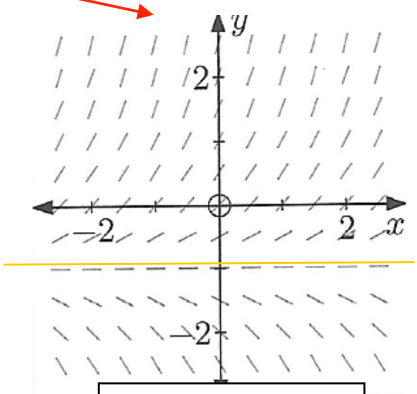
Question 3 **[3 marks]**

Match the slope fields to the differential equatons:

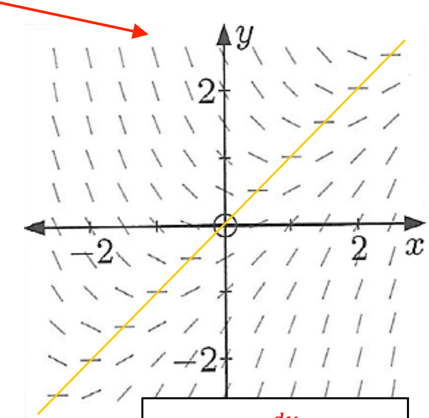
a) $\frac{dy}{dx} = y + 1$ ✓ b) $\frac{dy}{dx} = x - y$ ✓ c) $\frac{dy}{dx} = x - y^2$ ✓



If $y^2 = x, \frac{dy}{dx} = 0$



If $y = -1, \frac{dy}{dx} = 0$



If $y = x, \frac{dy}{dx} = 0$

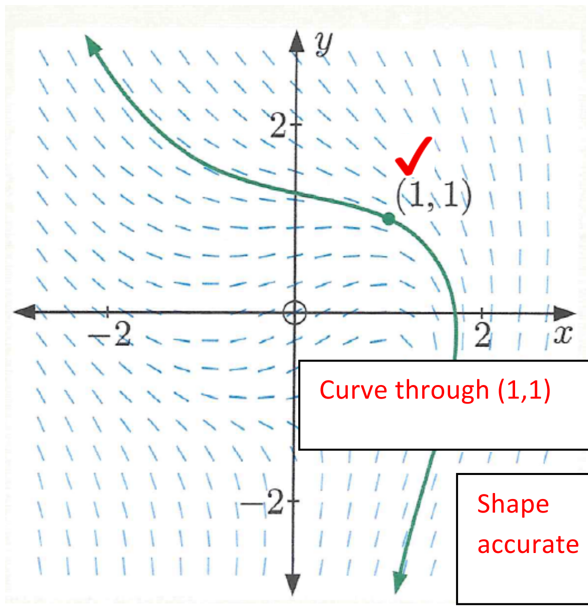
Question 4 **[1,2 = 3 marks]**

The slope field for $\frac{dy}{dx} = \frac{1-x^2-y^2}{y-x+2}$ is shown.

a) Find the gradient of the tangent to the solution curve at $(1,1)$.

At (1,1) $\frac{dy}{dx} = \frac{1-1-1}{1-1+2} = -\frac{1}{2}$ ✓

b) Sketch the solution curve which passes through $(1,1)$.



Curve through (1,1)

Shape accurate

Question 5 **[2,2 = 4 marks]**

The velocity of $v\ m/s$ of particle P is related to its displacement $x\ m$ from a fixed origin point O by the equation $v = x^3 - 8$. It is known that P starts off at O.

a) Determine the acceleration when $x = 1\ m$.

$$a = v\frac{dv}{dx} = (x^3 - 8)(3x^2)$$

When $x = 1$, $a = (1 - 8)(3) = -21 m/s^2$

b) Determine its displacement when its acceleration is 0.

$$a = (x^3 - 8)(3x^2) = 0$$

$$x = 0\ m \quad or \quad x = 2\ m$$

Question 6 **[2 marks]**

The cost function for a particular item is given by $\$c$ where $c = 450 + 0.5x^2$ and x is the number of such items produced.

Find the marginal cost for $x = 10$ and explain what this tells you about the cost of producing one more item at this level of production.

$$\frac{dc}{dx} = x \qquad \therefore Marginal\ cost\ when\ x = 10\ is\ \$10.$$

This means the cost of producing one more item when currently producing 10 items (i.e the cost of producing the 11th item) would be \$10.

Question 7 **[3,2,1,3 = 9 marks]**

An object is moving with simple harmonic motion. Its acceleration is given by $a = -4x\ ms^{-2}$, where x is the displacement. The object is initially at the origin and travelling with velocity $8\ ms^{-1}$.

a) Find the period and amplitude of the motion.

Simple harmonic motion		
If $\frac{d^2x}{dt^2} = -k^2x$ then	$x = A\sin(kt + \alpha)$	or $x = A\cos(kt + \beta)$
where A is the amplitude, α and β are phase angles, v is the velocity and x is the displacement		
$v^2 = k^2(A^2 - x^2)$	Period: $T = \frac{2\pi}{k}$	Frequency: $f = \frac{1}{T}$

$k = 2,\ \ Period\ = \frac{2\pi}{2} = \pi s$ ✓

$$x = A\sin(2t + \alpha)$$

$t = 0, x = 0 = A\sin\alpha \quad \Rightarrow \alpha = 0$ ✓ $\quad \Rightarrow \quad x = A\sin(2t)$

$$v = 2A\cos 2t$$

$t = 0, v = 8 = 2A\cos 0 \quad \Rightarrow A = 4$

∴ Amplitude is 4. ✓

Alternate Working:

$v^2 = k^2(A^2 - x^2)$

$8^2 = 4(A^2 - 0)$ ✓

$A^2 = 16$

$Amplitude = 4$ ✓

b) Find the velocity and acceleration of the object the first time it is 3 m to the right of the origin.

$a = -4(3) = -12ms^{-2}$ ✓

$$v^2 = 4(16 - 9) = 4(7)$$

$v = 2\sqrt{7}ms^{-1}$ ✓

Or

$$x = 3 = 4\sin(2t) \quad \Rightarrow \quad \sin(2t) = \frac{3}{4}$$

$$\cos(2t) = \frac{\sqrt{7}}{4}$$

$$v = 8\cos 2t = 2\sqrt{7}ms^{-1}$$

c) Write x in terms of t.

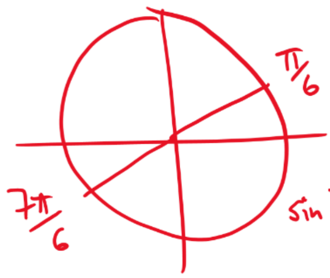
$$x = 4\sin(2t)$$ ✓

d) Find the position, velocity and acceleration of the object after $\frac{7\pi}{12}$ seconds.

$x = 4\sin\left(\frac{7\pi}{6}\right) = 4\left(-\frac{1}{2}\right) = -2$ i.e 2 m to the left of O. ✓

$v = 8\cos 2t = 8\left(-\frac{\sqrt{3}}{2}\right) = -4\sqrt{3}ms^{-1}$ ✓

$a = -k^2x = -4(-2) = 8ms^{-2}$ ✓



$\sin\frac{7\pi}{6} = -\sin\frac{\pi}{6} = -\frac{1}{2}$

$\cos\frac{7\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$

Year 12 Mathematics Specialist 2018

Test Number 5:

Rates of Change and Differential Equations

Resource Rich

Name: SOLUTIONS **Teacher: DDA**

Marks: **12**

Time Allowed: **15 minutes**

Instructions: You are permitted 1 A4 page of notes and your calculator. Show your working where appropriate remembering you must show working for questions worth more than 2 marks.

Question 8 **[4, 1, 3 = 8 marks]**

An ecology student is studying the repopulation of wild emus in South Australia after a drought. She notices that the population growth rate is approximately logistic so that $\frac{dP}{dt} = gP\left(1 - \frac{P}{A}\right)$ where $P(t)$ is the population t years after her study begins.

The carrying capacity A is known to be 2550 emus since this was the population before the drought.

Initially, the student finds that there are 310 emus. After two years she finds there are 390 emus.

a) Solve the differential equation, using the population $P(0)$ and $P(2)$, to find an expression for the population $P(t)$ after t years.

$$P = \frac{AP_0}{P_0 + (A - P_0)e^{-gt}}$$

$$P = \frac{2550 \times 310}{310 + (2550 - 310)e^{-gt}}$$ ✓

$$t = 2, P = 390 \quad \Rightarrow 390 = \frac{790500}{310 + 2240e^{-2g}}$$ ✓

$$\text{solve}\left(390 = \frac{790500}{310 + 2240e^{-2x}}\right)$$

{x=0.1329710429}

$$g = 0.133 \ (to\ 3\ dp)$$ ✓

$$\therefore P = \frac{790500}{310 + (2240)e^{-0.133t}}$$ ✓

Alternate:

$$\frac{dy}{dx} = \frac{g}{A}P(A - P)$$

$$P = \frac{2550 \times 310}{310 + (2550 - 310)e^{-\frac{g}{2550} \times 2550t}}$$

b) Estimate the emu population after 5 years.

$$\therefore P = \frac{790500}{310 + (2240)e^{-0.133(5)}} \approx 541\ Emus$$ ✓

$$\frac{790500}{310 + 2240e^{-0.133x}} \Big| x=5$$

540.7077703

c) Find the time at which the population growth rate, $\frac{dP}{dt}$, is a maximum.

$\frac{dP}{dt}$ *is* max *when* $\frac{d^2P}{dt^2} = 0$ *and* $\frac{d^3P}{dt^3} < 0$

$\frac{d^2P}{dt^2} = 0 \quad \rightarrow t = 14.8729702 \approx 14.9\ years$ ✓✓

$$\left.\frac{d^2P}{dt^2}\right|_{t=14.87297} = -0.749 \ldots < 0 \quad \therefore Max$$ ✓

$$\frac{d^2}{dx^2}\left(\frac{790500}{310+2240\times e^{-.13297x}}\right)$$

$$-\frac{1.215037302E\text{-}7\cdot(7.987839187E{+}13\cdot 2.718281828^{0.26594\cdot x}-5.771857993E{+}14\cdot 2.718281828^{0.13297\cdot x})}{(31\cdot 2.718281828^{0.13297\cdot x}+224)^3}$$

$$\text{solve}\left(-\frac{1.215037302E\text{-}7\cdot(7.987839187E{+}13\cdot 2.718281828^{0.26594\cdot x}-5.771857993E{+}14\cdot 2.718281828^{0.13297\cdot x})}{(31\cdot 2.718281828^{0.13297\cdot x}+224)^3}=0\right.$$

$\{x=14.8729702\}$

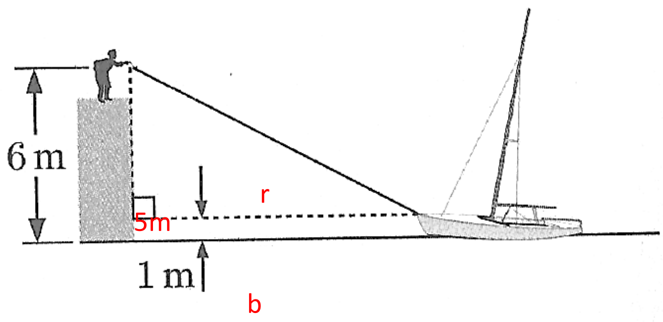
$$\frac{d^3}{dx^3}\left(\frac{790500}{310+2240\times e^{-.13297x}}\right)|x=14.8729702$$

-0.749395705

Question 9

[4 marks]

A man on a jetty pulls a boat towards him by hauling a rope at the rate of 20 metres per minute. The rope is attached to the boat 1 m above water level, and the man's hands are 6 m above the water level. How fast is the boat approaching the jetty at the instant when it is 15 m from the jetty? (Give answer to 1 dp)



$r^2 = 25 + b^2$, $\frac{dr}{dt} = -20m/min,$ $b = 15\ m,\ \frac{db}{dt} = ?$

$$b = 15\ m \quad \Rightarrow \quad r = \sqrt{25 + 225} = \sqrt{250} = 5\sqrt{10}$$

$$2r.\frac{dr}{dt} = 2b.\frac{db}{dt}$$

$$2(5\sqrt{10})(-20) = 2(15)\frac{db}{dt}$$

$$\frac{db}{dt} = -\frac{20\sqrt{10}}{3} m/min = -21.08185 \ldots$$

Therefore the boat is approaching the jetty at the rate of 21.1m/min when it is 15m from the jetty.